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Solution of non-polynomial linear Volterra Integral Equations of the second type

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Abstract.

In this paper, we will study the linear Volterra integral equations of the second type, which are not polynomial (including linear and quadratic), Likewise, the linear Volterra's integral equations of the first type, and then transformed into the integral Volterra equations of the second type. Moreover, many examples are presented to clarify the accuracy, efficiency and ease of performance of the proposed method on the on hand.

Keywords: weakly singular kernel, constant parameter, homogenous, non-homogenous, degenerate or sparable, improper if, metric space.

1. Introduction.

Integrative equations play an important role in many of the orifical and applied research then to the possibility of expressing the integral equation as a continuous or non-continuous integral generator. Hence, we see that integrative equations play a fundamental role for mathematical modeling with complementary effects. In the science of mechanical applications, we find many elasticity issues. This is for the spreadable bodies that have flexible non-linear behavior. Viscosity with long memory can be expressed by the integral Volterra equation. In general, there are other applications in the applied field.

The study in this paper is divided as follows: In section (1.2), we study the classification of integral equations, and some basic concepts are given. In section (1.3), the mathematical theory of the existence and uniqueness theorem for linear VIE's will be considered. In section (1.4), some analytical methods are considered to find the solution of linear VIE's of the second kind and VIE's with weakly singular kernel.

We will mention some basic definitions for integral equations.

(1.1) Definition.[10].

An integral equation is that equation in which the unknown function $u(x)$ appears inside an integral sign. The most standard type of integral equation in $u(x)$ is of the form:

$$h(x)u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)u(t)dt \quad , x \in [a,b] \quad (1.1)$$

where $a(x)$ and $b(x)$are the limits of integration, λ is a constant parameter, and $k(x,t)$ is a known function of two variables x and t, which are called the kernel of the integral equation. The functions $f(x)$ and $k(x,t)$ are given in advance. It is to be noted that the limits of integration determined as $a(x)$and $b(x)$ and may be both variables, constants, or mixed.

(1.2) Definition.[5].

An integral equation (1.1) is called non-linear integral equation, if the kernel $k(x,t)$is given in the form $k(x,t,u(t))$.

(1.3) Definition.[2].

The linear integral equation (1.1) is called homogenous, if $f(x) = 0$, otherwise it is called non- homogenous.

(1.4) Definition.[7].

The equation (1.1) is called linear integral equation of the first kind, if $h(x) = 0$, while if $h(x) = 1$, it called linear integral equation of the second kind, otherwise it is called of the third kind.

(1.5) Definition.[9].

The integral equation is called Volterra integral equation, when $a(x) = a$ and $b(x) = x$, where a is constant that is:

$$h(x)u(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt \quad ,x \in [a,b] \quad (1.2)$$

(1.6) Definition.[2].

The integral equation is called Fredholm integral equation, when $a(x) = a$ and $b(x) = b$, where a and b are constant, which has a form:

$$h(x)u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t)dt \quad ,x \in [a,b] \quad (1.3)$$

(1.7) Definition.[5].

If the kernel in integral equation (1.1) depends on the difference $(x - t)$, then it is called difference kernel and the equation is called integral equation of convolution type.

i.e., $k(x,t) = k(x - t)$.

Here we can apply Laplace transform to get the exact solution.

(1.8) Definition.[5].

The kernel is called degenerate or (sparable) kernel, when the kernel may be decomposed as follows:

$$k(x,t) = \sum_{k=1}^{n} a_k(x) b_k(x)$$

(1.9) Definition.[10].

An integral differential equation is an equation involving derivative and integral together with unknown function $u(x)$ which is of the form:

$$u^{(k)}(x) + \sum_{j=0}^{k-1} p_j(x) u^j(x) = f(x) + \int_{a(x)}^{b(x)} k(x,t) u(t) dt \quad (1.4)$$

Where $u^{(j)}(x) = \frac{d^j u}{dx^j}$

(1.10) Definition.[9].

The integral $\int_a^b f(x)dx$ is called improper if.

(i) $a = \infty \; or \; b = \infty$ or both

(ii) $f(x)$is unbounded at one or more points of $a \leq x \leq b$ (there points are called singular points). Moreover, it is called singular if the kernel $k(x,t)$ becomes unbounded at one or more points in the interval of integration.

*Integral corresponds to (i) and (ii) are called improper integrals of the $1^{st}$and 2^{nd} kind respectively.

* Integral with both (i) and (ii) are called improper integrals of the 3^{rd} kind.

(1.11) Definition.[9].

If the kernel $k(x,t)$ is in the form $k(x,t) = \frac{H(x,t)}{(x-\epsilon)^{\alpha}}$.

Where H is bounded in D: $a \le x \le b \; and \; a \le t \le b$ with $H(x,t) \neq 0$ and α is constant s.t $0 \le \alpha \le 1$ then the integral equation is called weakly singular. The equations of the form:

$$f(x) = \int_{0}^{x} \frac{u(t)}{(x-t)^{\alpha}} dt \qquad 0 < \alpha < 1 \qquad (1.5)$$

or of the second kind

$$u(x) = f(x) + \int_{0}^{x} \frac{u(t)}{(x-t)^{\alpha}} dt \qquad 0 < \alpha < 1 \qquad (1.6)$$

are called generalized Abel's integral equation and weakly singular integral equations respectively. For $\alpha = \frac{1}{2}$

$$f(x) = \int_{0}^{x} \frac{u(t)}{(x-t)^{\frac{1}{2}}} dt$$

This is called the Abel's singular integral equation. We will focus our concern on equation of the form:

$$u(x) - \int_{0}^{t} \frac{t^{\mu-1}}{x^{\mu}} u(t) dt = f(x) \qquad , \qquad x \in [0,T]$$

This is can be classified as Volterra integral equations of the second kind with weakly singular kernel. Where $u(t)$ is unknown function and f is known function, where $0 < \mu < 1$. However, there is a singularity at $t = 0$ and $s = 0$ for any positive value of t.

In this paper we will consider the two following problems.

* Linear Volterra integral equation of the Second kind (VIE's) with $\lambda = 1$, of the form:

$$u(x) = f(x) + \int_{a}^{x} k(x,t)u(t)dt \qquad (1.7)$$

* Linear Volterra integral equations of the Second kind with weakly singular kernel, of the form:

$$u(x) - \int_{a}^{x} \frac{t^{\mu-1}}{x^{\mu}} u(t)dt = f(x) \quad , \quad x \in [0,T] \qquad 1.8)$$

2. Existence and Uniqueness.

In this section, we will try to impose a certain condition in order to prove the existence and uniqueness theorem for integral equation to be applied to linear VIE's of the second kind. Before we prove existence and uniqueness, some definitions are presented; A background and review will be needed to prove the main results of this section.

(2.1) Definition.[1].

Let $\{f_n(t)\}$ be a sequence of functions from an interval $[a,b]$ to real numbers, then:

* $\{f_n(t)\}$ is uniformly bounded on $[a,b]$ if there exists M such that n a positive integer and $t \in [a,b]$ $imply$ $|f_n(t)| \leq M$.

* $\{f_n(t)\}$ is equicontinuous if for any $\in > 0$ there exists $\delta > 0$, such that:

$[n\ is\ a\ positive\ integer, t_1, t_2 \in [a,b]\ and\ |t_1 - t_2| < \delta]$

$imply\ |f_n(t_1) - f_n(t_2)| < \epsilon.$

(2.1.1) Theorem.[1].

Let $(t_0, x_0) \in R^{n+1}$ and suppose there are positive constants a, b and M, such that $D = \{(t,x): |t - t_0| \leq b\}, G: D \to R^n$ is continuous, and $|G(t,x)| \leq M, if\ (t,x) \in D.$

Then there is at least one solution

$$x(t)\ of: \acute{x} = G(t,x), x(t_0) = x_0 \qquad (1.9)$$

and $x(t)$ is define for $|t - t_0| \leq T\ with\ T = \min\{a, b, M\}$.

(2.2) Definition.[8].

Let $U \subset R^{n+1}$ and $G: U \to R^{n+1}$ we say that G satisfies a local lipscitz condition with respect to x if for each compact subset M of U there is a

constant k such that $(t, x_1) and\ (t, x_2)$ in M implies:

$$|G(t,x_1) - G(t,x_2)| \leq K|x_1 - x_2| \qquad (1.10)$$

(2.2.1) Theorem.[1].

Let the conditions of theorem (1.1) hold and suppose that there is a constant L such that for all $(t,x_1), (t,x_2) \in D$ implies:

$$|G(t,x_1) - G(t,x_2)| \leq L|x_1 - x_2|$$

Then (1.9) has only one solution.

(2.3) Definition.[8].

A pair $(\mathcal{L}, p)$ is a metric space if $\mathcal{L}$ is a non-empty set and $p: \mathcal{L} \times \mathcal{L} \to [0, \infty)$ such that when y, z and u are in $\mathcal{L}$, then:

a) $p(y,z) \geq 0\ and\ p(y,y) = 0.$

b) $p(y,z) = p(z,y)$.

c) $p(y,z) \leq p(y,u) + p(u,z)$.

(2.4) Definition.[1].

Let $(\mathcal{L}, p)$ be a metric space and $A: \mathcal{L} \to \mathcal{L}$ the operator A is a contraction operator if there is an $\alpha \in (0,1)$ such that:

$$x \in \mathcal{L} \; and \; y \in \mathcal{L} \; imply \; p[A(x), A(y)] \leq \alpha p(x,y)$$

(2.4.1) Theorem. (contractive mapping principle) [1].

Let $(\mathcal{L}, p)$ be a complete metric space and $A: \mathcal{L} \to \mathcal{L}$ a contraction operator. Then there is a unique with $\emptyset \in \mathcal{L} \; with \; A(\emptyset) = \emptyset$.

(2.4.2) Theorem.[1].

Let a, b and L be positive number, and for some fixed $\alpha \in (0,1)$, define $c = \alpha b$ suppose:

a) f is continuous on $[0, a]$, also integrable and bonded and satisfy Lipshitz condition.

b) K is continuous on $U = \{(t,s,x) : 0 \leq s, t \leq \alpha \; and \; |x - f(t)| \leq b\}$.

c) K satisfies Lipshitz condition with respect to x on U

$$|K(t,s,x) - K(t,s,y)| \leq L|x - y|$$

$If \;\; (t,s,x), (t,s,y) \in U, If \;\; M = t^{max}|K(t,s,x)|$

then there is a unique solution of:

$$u(t) = f(t) + \int_0^t K\big(t,s,u(s)\big)ds \;\; on \; [0,T], where \; T = \min\{a, \frac{b}{M}, c\}$$

3. Analytical Methods for Solving VIE's.

In this section, some methods which have been used for solving linear VIE's of the second kind and VIE'S with weakly singular kernel have been studied and illustrated by examples.

(3.1) Solution of Linear VIE's of the Second Kind.[10].

We will first define Volterra integral equations of the second kind given by:

$$u(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt \qquad ,a \leq x \leq b$$

The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given continues functions.

(3.2) A domain Decomposition Method.[10].

The A domain decomposition method (ADM) was introduced and developed by George A domain. The A domain decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (1.11)$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

where the components $u_n(x), n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components individually; we substitute (1.11) into the Volterra integral equation to obtain.

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_0^x k(x,t)\Big(\sum_{n=0}^{\infty} u_n(t)\Big)dt \qquad (1.12)$$

The zeroth component $u_0(x)$is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x), j \geq 1$ of the unknown function $u(x)$is completely determined by setting the recurrence relation:

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \int_0^x k(x,t)u_n(t)dt\,, \quad n \geq 0 \qquad (1.13)$$

Example (3.2.1).[10].

To Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t)dt \qquad (1.14)$$

where

$$f(x) = 1 \; and \; k(x,t) = -1$$

Substituting decomposition series (1.11) in to both side of VIE (1.14) gives,

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)dt$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1,$$

$$u_{k+1}(x) = -\int_0^x u_k(t)dt\,, \qquad k \geq 0$$

so that

$$u_0(x) = 1,$$

$$u_1(x) = -\int_0^x u_0(t)dt = -\int_0^x 1dt = -x\,,$$

$$u_2(x) = -\int_0^x u_1(t)dt = -\int_0^x -tdt = \frac{x^2}{2!},$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x \frac{t^2}{2!}dt = -\frac{x^3}{3!},$$

$$u_4(x) = -\int_0^x u_3(t)dt = -\int_0^x -\frac{t^3}{3!}dt = \frac{x^4}{4!},$$

And so on. Gives the series solution

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^{-x}$$

Which is the exact solution for equation (1.14).

Example (3.2.2).

To Solve the following Volterra integral equation:

$$u(x) = 1 + \int_0^x u(t)dt \qquad (1.15)$$

Consider that

$$u_0(x) = 0$$

As $u_0(x) = 0$, then $u_1(x) = 1$, and therefore.

$$u_2(x) = 1 + \int_0^x u_1(t)dt = 1 + \int_0^x 1dt = 1 + x\,,$$

$$u_3(x) = 1 + \int_0^x u_2(t)dt = \ 1 + \int_0^x (1+t)dt = 1 + x\frac{x^2}{2}\,,$$

$$u_4(x) = 1 + \int_0^x u_3(t)dt = \ 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right)dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\,,$$

And so on. Gives the series solutions.

$$u(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Which is the exact solution for equation (1.15).

(3.3) The Successive Approximations Method. [2, 10].

The successive approximations method, also called the Picard iteration method. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations. The successive approximations method introduces the recurrence relation

$$u_n(x) = f(x) + \int_a^x k(x,t)u_{n-1}(t)dt \quad ,n \geq 1 \qquad (1.16)$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$, mostly we select $0,1,x$ for $u_0(x)$ and by using (1.16), several successive approximations $u_k(x), k \geq 1$ will be determined as:

$$u_1(x) = f(x) + \int_a^x k(x,t)u_0(t)dt$$

$$u_2(x) = f(x) + \int_a^x k(x,t)u_1(t)dt$$

$$u_3(x) = f(x) + \int_a^x k(x,t)u_2(t)dt$$

.

$$u_n(x) = f(x) + \int_a^x k(x,t)u_{n-1}(t)dt$$

The successive approximations method or the Picard iteration method will be illustrated by the following example.

Example (3.3.1).[10].

To solve the following Volterra integral equation by using the successive approximations method,

$$u(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x tu(t)dt \qquad (1.17)$$

For the zeroth approximation $u_0(x)$ we select $u_0(x) = 0$, We next use the iteration formula

$$u_{n+1}(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x tu_n(t)dt \ , n \geq 0 \qquad (1.18)$$

Substituting $u_0(x)$ in equation (1.18), we obtain

$$u_1(x) = -1 + e^x + \frac{1}{2}x^2e^x$$

$$u_2(x) = -3 + \frac{1}{4}x^2 + e^x\left(3 - 2x + \frac{5}{4}x^2 - \frac{1}{4}x^3\right),$$

$$u_3(x) = x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right),$$

$$u_{n+1}(x) = x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = xe^{-x}$$

Which is the exact solution for equation (1.17).

(3.4) The Laplace Transformation Method. [2,5,10].

The Laplace transformation method can be used for solving integral equation, It was stated that if the kernel depends on the difference $(x - t)$. Then by taking Laplace transform for both sides of VIE's we find:

$$U(s) = F(s) + K(s)U(s) \qquad (1.19)$$

Where $U(s) = L\{u(x)\}, K(s) = L\{K(x)\}, F(s) = L\{f(x)\}$.

The solution of $u(x)$ is obtained by taking the invers of Laplace transform of

$$U(s) = \frac{F(s)}{1 - K(s)}, \qquad K(s) \neq 0$$

Then we find

$$u(x) = L^{-1}\{\frac{F(s)}{1 - K(s)}\}$$

This method will be illustrated by example (1.3).

Example (3.4.1).[10].

To solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \qquad (1.20)$$

Where $f(x) = 1$ *and* $k(x,t) = (x-t)$, Taking Laplace transforms of both sides of equation (1.20) gives:

$$L\{u(x)\} = L(1) - L\{(x)L(u)\}$$

So that

$$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s)$$

$$U(s) = \frac{s}{1+s^2}$$

By taking the invers of Laplace transform, of $U(s)$, we obtain that $u(x) = Cosx$, which is the exact solution for equation (1.20).

Example (3.4.2).

We wish to solve the Volterra integral equation:

$$u(x) = 1 + \int_0^x u(t)dt \qquad (1.21)$$

Using the Laplace transform method.

Notice that the kernel $k(x,t) = 1, \lambda = 1$. Taking Laplace transforms of both sides gives:

$$U(s) = \frac{1}{s} + \frac{1}{s}U(s)$$

$$U(s) = \frac{1}{s-1}$$

Now by taking the invers Laplace transform of both sides, the exact solution is found to be given by $u(x) = e^x$.

Example (3.4.3).

We wish to solve the Volterra integral equation:

$$u(x) = \sin(x) + \cos(x) + 2\int_0^x \sin(x-t)u(t)dt \quad (1.22)$$

Using the Laplace transform method.

We should use the linear property of the Laplace transforms here. Taking Laplace transforms of both sides gives:

$$U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}U(s)$$

Or equivalently

$$U(s) = \frac{1}{s-1}$$

Now by taking the invers Laplace transform of both sides, the exact solution is found to be given by $u(x) = e^x$.

Example (3.4.4).

We wish to solve the Volterra integral equation:

$$u(x) = \frac{x^3}{6} - \int_0^x (x-t)u(t)dt \qquad (1.23)$$

Using the Laplace transform method.

Taking the Laplace transforms of both sides gives:

$$U(s)=\frac{1}{6}\times\frac{6}{s^4}-\frac{1}{s^2}U(s)$$

Or equivalently

$$U(s)=\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}$$

Now by taking the invers Laplace transform of both sides, the exact solution is found to be given by $u(x)=x-\sin{(x)}$.

(3.5) Solution of Linear VIE's of the Second kind with weakly singular kernel. [3,4,6].

We consider the second kind VIE's with weakly singular kernel

$$u(x)-\int_0^t\frac{t^{\mu-1}}{x^\mu}u(t)dt=f(x),\qquad x\in[0,T]$$

where $0<\mu<1$ and f are known functions. However, there is a singularity at $t=0$ *and* $s=0$ for any positive value of t.

(3.6) Analytic Method.

In [3] the author gives suggestion for the analytic solution to solve linear VIE's of the second kind with weakly singular kernel.

(3.6.1) Lemma.[3].

(a): If $0<\mu<1$ and $f\in C^1[0,t](with\ f(0)=0\ if\ \mu=1)$ then VIE's of the second kind with weakly singular kernel (1.8), has a family of solution $u\in[0,t]$.

$$u(t) = C_0 t^{1-\mu} + f(t) + \gamma + t^{1-u} \int_0^t S^{\mu-2}\big(f(s) - f(0)\big)ds \quad (1.24)$$

Were.

$$\gamma = \begin{cases} \dfrac{1}{\mu - 1} f(0) & if\ \mu < 1 \\ 0 & if\ \mu = 1 \end{cases}$$

And C_0 is an arbitrary constant. Out of family of solutions there is one particular solution $u \in C^1[0, t]$. Such a solution is unique and can be obtained from (1.24) by taking $C_0 = 0$.

(b): if $\mu > 1\ and\ f \in C^m[0, t], m \geq 0,$ then the unique solution $u \in C^m[0, t]$ is:

$$u(t) = f(t) + t^{1-\mu} \int_0^t S^{\mu-2} f(s) ds \qquad (1.25)$$

We note that (1.21) can be obtained from (1.20) with Indeed, from it follows (1.24) that

$$C_0 = \lim_{t \to 0^+} t^{\mu-1} u(t)$$

and this limit is zero when $\mu > 1$. In principle, if we know the value of C_0 we may use (1.24) to obtain the numerical approximations of the solution.

4. Conclusions.

In this paper, a simple review of Volterra's integral equations was presented, especially in section (1-3), after that, his analysis method was adopted to solve linear Volterra integral equations including the method of (A domain) analysis, the successive approximation method, and the Laplace transform method for solving Volterra integral equation with weakly singular kernel.

-1- Many problems can be solved using analytical methods.

-2- There are new ways to solve Volterra's integral equations, because there is no single method that works well for all of these equations.

5. Recommendations.

In this paper, we recommend researchers that there are other types of methods that can be used in solving linear Volterra integral equations, which is the numerical method of treating these equations numerically in the same using (math lab) programs when a function is given in tabular form, which is the introduction of a numerical method using functions. There are not multiple boundaries, to solve these linear Volterra integral equations of the first and second types with weakly singular kernel.

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